CHAPTER 5

Motion in Two Dimensions

Figure 5.1 Billiard balls on a pool table are in motion after being hit with a cue stick. (Popperipopp, Wikimedia Commons)

Chapter Outline

INTRODUCTION In Chapter 2, we learned to distinguish between vectors and scalars; the difference being that a vector has magnitude and direction, whereas a scalar has only magnitude. We learned how to deal with vectors in physics by working straightforward one-dimensional vector problems, which may be treated mathematically in the same as scalars. In this chapter, we'll use vectors to expand our understanding of forces and motion into two dimensions. Most real-world physics problems (such as with the game of pool pictured here) are, after all, either two- or three-dimensional problems and physics is most useful when applied to real physical scenarios. We start by learning the practical skills of graphically adding and subtracting vectors (by using drawings) and analytically (with math). Once we're able to work with two-dimensional vectors, we apply these skills to problems of projectile motion, inclined planes, and harmonic motion.

5.1 Vector Addition and Subtraction: Graphical Methods

Section Learning Objectives

By the end of this section, you will be able to do the following:

- Describe the graphical method of vector addition and subtraction
- Use the graphical method of vector addition and subtraction to solve physics problems

Section Key Terms

graphical method	head (of a vector)	head-to-tail method	resultant
resultant vector	tail	vector addition	vector subtraction

The Graphical Method of Vector Addition and Subtraction

Recall that a vector is a quantity that has magnitude and direction. For example, displacement, velocity, acceleration, and force are all vectors. In one-dimensional or straight-line motion, the direction of a vector can be given simply by a plus or minus sign. Motion that is forward, to the right, or upward is usually considered to be *positive* (+); and motion that is backward, to the left, or downward is usually considered to be *negative* (–).

In two dimensions, a vector describes motion in two perpendicular directions, such as vertical and horizontal. For vertical and horizontal motion, each vector is made up of vertical and horizontal components. In a one-dimensional problem, one of the components simply has a value of zero. For two-dimensional vectors, we work with vectors by using a frame of reference such as a coordinate system. Just as with one-dimensional vectors, we graphically represent vectors with an arrow having a length proportional to the vector's magnitude and pointing in the direction that the vector points.

Figure 5.2 shows a graphical representation of a vector; the total displacement for a person walking in a city. The person first walks nine blocks east and then five blocks north. Her total displacement does not match her path to her final destination. The displacement simply connects her starting point with her ending point using a straight line, which is the shortest distance. We use the notation that a boldface symbol, such as **D**, stands for a vector. Its magnitude is represented by the symbol in italics, D, and its direction is given by an angle represented by the symbol θ. Note that her displacement would be the same if she had begun by first walking five blocks north and then walking nine blocks east.

TIPS FOR SUCCESS

In this text, we represent a vector with a boldface variable. For example, we represent a force with the vector **F**, which has both magnitude and direction. The magnitude of the vector is represented by the variable in italics, F, and the direction of the variable is given by the angle θ.

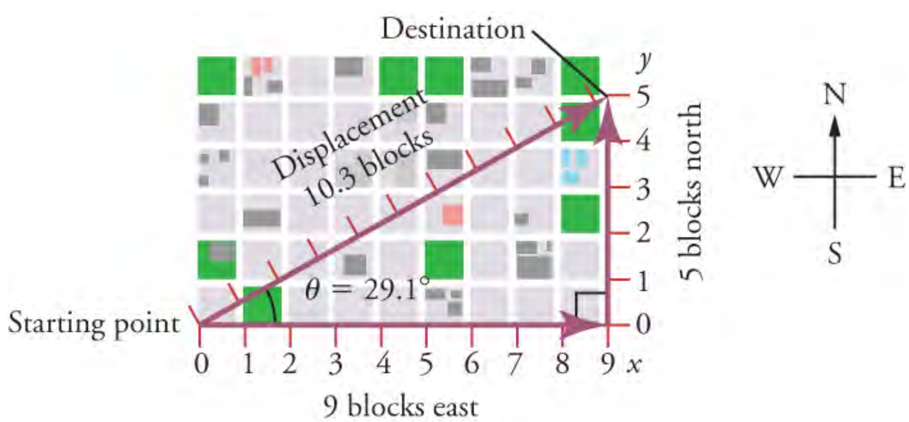

Figure 5.2 A person walks nine blocks east and five blocks north. The displacement is 10.3 blocks at an angle 29.1° north of east.

The **head-to-tail method** is a **graphical** way to add vectors. The **tail** of the vector is the starting point of the vector, and the **head** (or tip) of a vector is the pointed end of the arrow. The following steps describe how to use the head-to-tail method for graphical **vector addition**.

1. Let the x-axis represent the east-west direction. Using a ruler and protractor, draw an arrow to represent the first vector (nine blocks to the east), as shown in Figure 5.3(a).

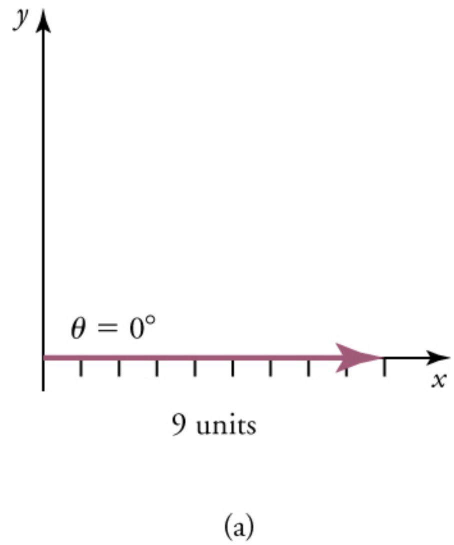

Figure 5.3 The diagram shows a vector with a magnitude of nine units and a direction of 0°.

2. Let the y-axis represent the north-south direction. Draw an arrow to represent the second vector (five blocks to the north). Place the tail of the second vector at the head of the first vector, as shown in Figure 5.4(b).

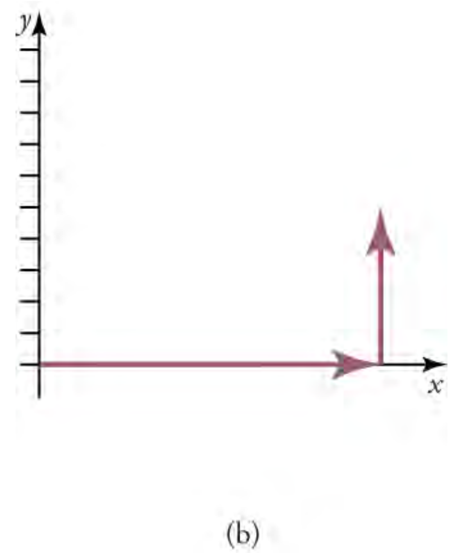

Figure 5.4 A vertical vector is added.

3. If there are more than two vectors, continue to add the vectors head-to-tail as described in step 2. In this example, we have only two vectors, so we have finished placing arrows tip to tail.
4. Draw an arrow from the tail of the first vector to the head of the last vector, as shown in Figure 5.5(c). This is the **resultant**, or the sum, of the vectors.

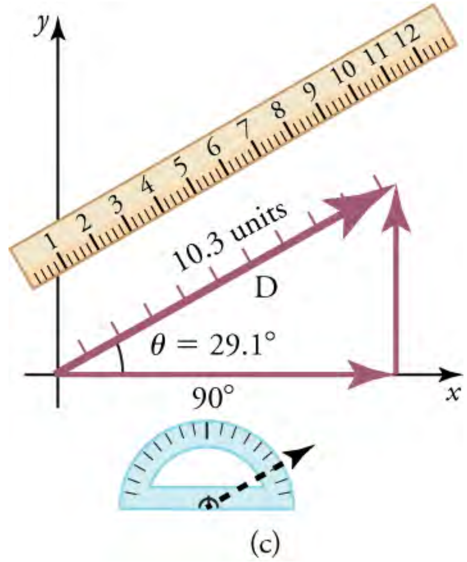

Figure 5.5 The diagram shows the resultant vector, a ruler, and protractor.

5. To find the magnitude of the resultant, measure its length with a ruler. When we deal with vectors analytically in the next section, the magnitude will be calculated by using the Pythagorean theorem.
6. To find the direction of the resultant, use a protractor to measure the angle it makes with the reference direction (in this case, the *x*-axis). When we deal with vectors analytically in the next section, the direction will be calculated by using trigonometry to find the angle.

WATCH PHYSICS

Visualizing Vector Addition Examples

This video shows four graphical representations of vector addition and matches them to the correct vector addition formula.

[Click to view content (https://openstax.org/l/02addvector)](https://openstax.org/l/02addvector)

GRASP CHECK

There are two vectors $\vec{a}$ and $\vec{b}$. The head of vector $\vec{a}$ touches the tail of vector $\vec{b}$. The addition of vectors $\vec{a}$ and $\vec{b}$ gives a resultant vector $\vec{c}$. Can the addition of these two vectors can be represented by the following two equations? $\vec{a} + \vec{b} = \vec{c}$; $\vec{b} + \vec{a} = \vec{c}$

a. Yes, if we add the same two vectors in a different order it will still give the same resultant vector.
b. No, the resultant vector will change if we add the same vectors in a different order.

Vector subtraction is done in the same way as vector addition with one small change. We add the first vector to the negative of the vector that needs to be subtracted. A negative vector has the same magnitude as the original vector, but points in the opposite direction (as shown in Figure 5.6). Subtracting the vector **B** from the vector **A**, which is written as **A** – **B**, is the same as **A** + (–**B**). Since it does not matter in what order vectors are added, **A** – **B** is also equal to (–**B**) + **A**. This is true for scalars as well as vectors. For example, 5 – 2 = 5 + (–2) = (–2) + 5.

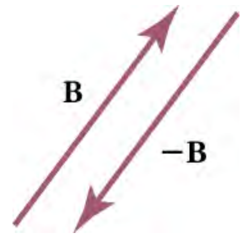

Figure 5.6 The diagram shows a vector, B, and the negative of this vector, –B.

Global angles are calculated in the counterclockwise direction. The clockwise direction is considered negative. For example, an angle of 30° south of west is the same as the global angle 210°, which can also be expressed as -150° from the positive x-axis.

Using the Graphical Method of Vector Addition and Subtraction to Solve Physics Problems

Now that we have the skills to work with vectors in two dimensions, we can apply vector addition to graphically determine the **resultant vector**, which represents the total force. Consider an example of force involving two ice skaters pushing a third as seen in Figure 5.7.

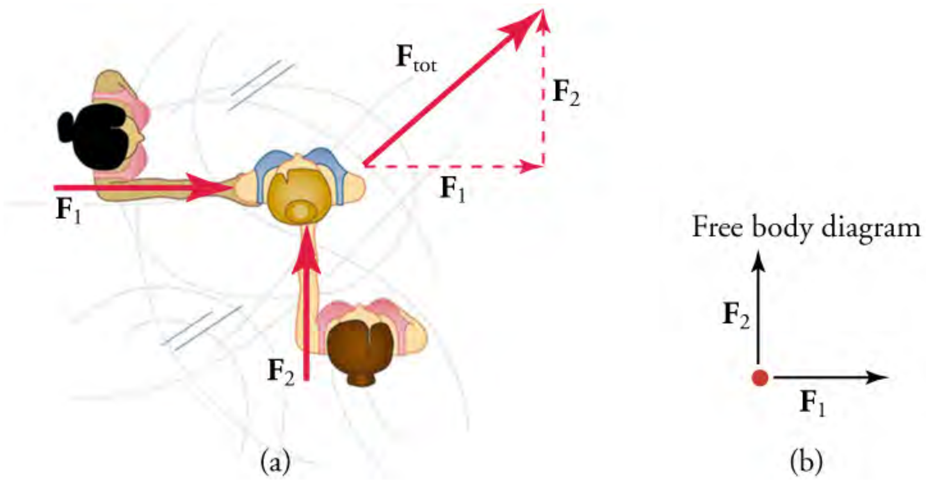

Figure 5.7 Part (a) shows an overhead view of two ice skaters pushing on a third. Forces are vectors and add like vectors, so the total force on the third skater is in the direction shown. In part (b), we see a free-body diagram representing the forces acting on the third skater.

In problems where variables such as force are already known, the forces can be represented by making the length of the vectors proportional to the magnitudes of the forces. For this, you need to create a scale. For example, each centimeter of vector length could represent 50 N worth of force. Once you have the initial vectors drawn to scale, you can then use the head-to-tail method to draw the resultant vector. The length of the resultant can then be measured and converted back to the original units using the scale you created.

You can tell by looking at the vectors in the free-body diagram in Figure 5.7 that the two skaters are pushing on the third skater with equal-magnitude forces, since the length of their force vectors are the same. Note, however, that the forces are not equal because they act in different directions. If, for example, each force had a magnitude of 400 N, then we would find the magnitude of the total external force acting on the third skater by finding the magnitude of the resultant vector. Since the forces act at a right angle to one another, we can use the Pythagorean theorem. For a triangle with sides a, b, and c, the Pythagorean theorem tells us that

$$a^2 + b^2 = c^2$$
$$c = \sqrt{a^2 + b^2}.$$

Applying this theorem to the triangle made by $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_{\text{tot}}$ in Figure 5.7, we get

$$\mathbf{F}_{\text{tot}}^2 = \sqrt{\mathbf{F}_1^2 + \mathbf{F}_1^2},$$

or

$$\mathbf{F}_{\text{tot}} = \sqrt{(400\text{ N})^2 + (400\text{ N})^2} = 566\text{ N}.$$

Note that, if the vectors were not at a right angle to each other (90° to one another), we would not be able to use the Pythagorean theorem to find the magnitude of the resultant vector. Another scenario where adding two-dimensional vectors is necessary is for velocity, where the direction may not be purely east-west or north-south, but some combination of these two directions. In the next section, we cover how to solve this type of problem analytically. For now let's consider the problem graphically.

WORKED EXAMPLE

Adding Vectors Graphically by Using the Head-to-Tail Method: A Woman Takes a Walk

Use the graphical technique for adding vectors to find the total displacement of a person who walks the following three paths (displacements) on a flat field. First, he walks 25 m in a direction 49° north of east. Then, he walks 23 m heading 15° north of east. Finally, he turns and walks 32 m in a direction 68° south of east.

Strategy

Graphically represent each displacement vector with an arrow, labeling the first **A**, the second **B**, and the third **C**. Make the lengths proportional to the distance of the given displacement and orient the arrows as specified relative to an east-west line. Use the head-to-tail method outlined above to determine the magnitude and direction of the resultant displacement, which we'll call **R**.

Solution

(1) Draw the three displacement vectors, creating a convenient scale (such as 1 cm of vector length on paper equals 1 m in the problem), as shown in Figure 5.8.

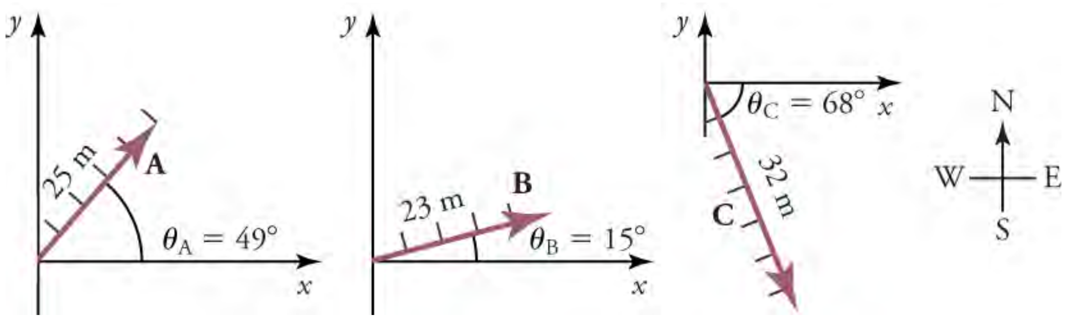

Figure 5.8 The three displacement vectors are drawn first.

(2) Place the vectors head to tail, making sure not to change their magnitude or direction, as shown in Figure 5.9.

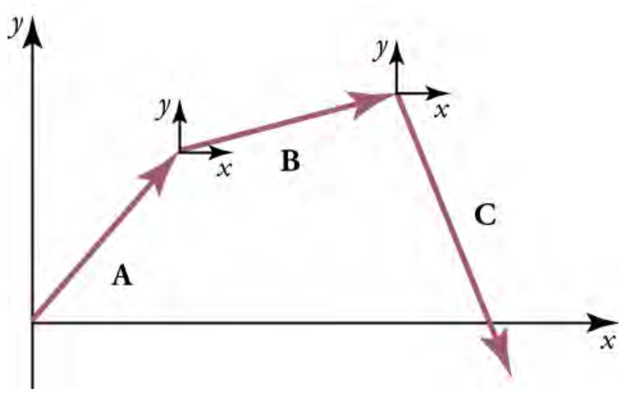

Figure 5.9 Next, the vectors are placed head to tail.

(3) Draw the **resultant vector R** from the tail of the first vector to the head of the last vector, as shown in Figure 5.10.

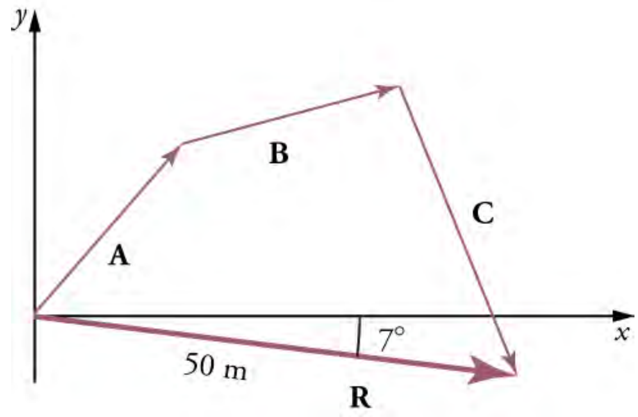

Figure 5.10 The resultant vector is drawn .

(4) Use a ruler to measure the magnitude of **R**, remembering to convert back to the units of meters using the scale. Use a protractor to measure the direction of **R**. While the direction of the vector can be specified in many ways, the easiest way is to measure the angle between the vector and the nearest horizontal or vertical axis. Since **R** is south of the eastward pointing axis (the *x*-axis), we flip the protractor upside down and measure the angle between the eastward axis and the vector, as illustrated in Figure 5.11.

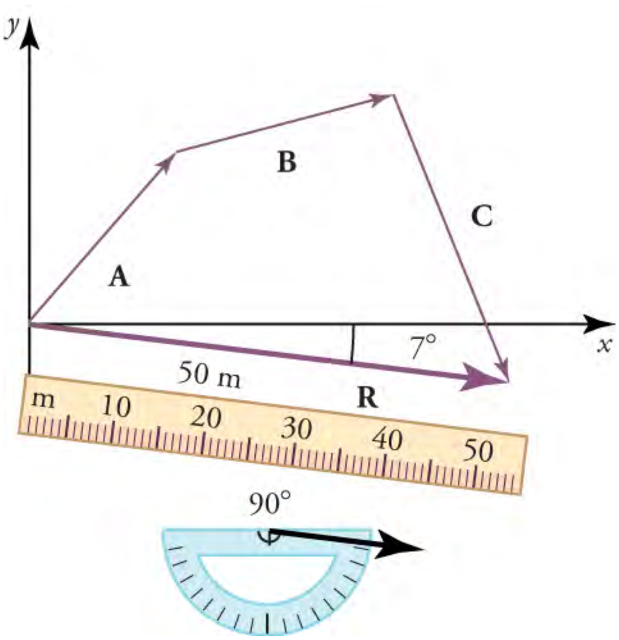

Figure 5.11 A ruler is used to measure the magnitude of **R**, and a protractor is used to measure the direction of **R**.

In this case, the total displacement **R** has a magnitude of 50 m and points 7° south of east. Using its magnitude and direction, this vector can be expressed as

$$\mathrm{R} = 50 \ \ \mathrm{m} \tag{5.1}$$

and

$$\theta = 7^\circ \text{south of east.} \tag{5.2}$$

Discussion

The head-to-tail graphical method of vector addition works for any number of vectors. It is also important to note that it does not matter in what order the vectors are added. Changing the order does not change the resultant. For example, we could add the vectors as shown in Figure 5.12, and we would still get the same solution.

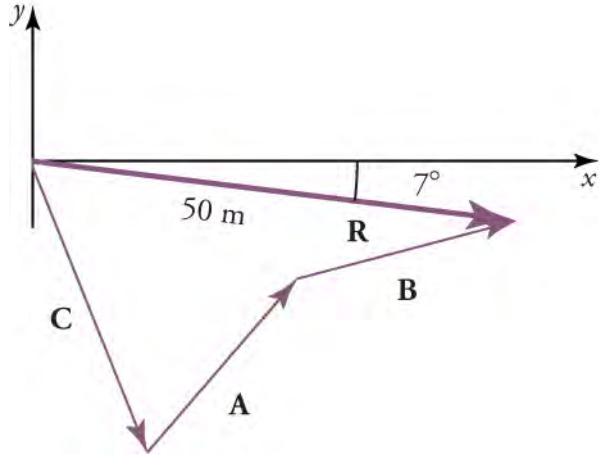

Figure 5.12 Vectors can be added in any order to get the same result.

WORKED EXAMPLE

Subtracting Vectors Graphically: A Woman Sailing a Boat

A woman sailing a boat at night is following directions to a dock. The instructions read to first sail 27.5 m in a direction 66.0° north of east from her current location, and then travel 30.0 m in a direction 112° north of east (or 22.0° west of north). If the woman makes a mistake and travels in the *opposite* direction for the second leg of the trip, where will she end up? The two legs of the woman's trip are illustrated in Figure 5.13.

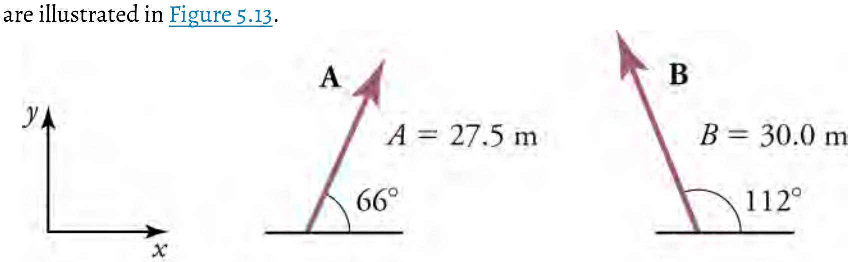

Figure 5.13 In the diagram, the first leg of the trip is represented by vector A and the second leg is represented by vector B.

Strategy

We can represent the first leg of the trip with a vector **A**, and the second leg of the trip that she was *supposed to* take with a vector **B**. Since the woman mistakenly travels in the opposite direction for the second leg of the journey, the vector for second leg of the trip she *actually* takes is –**B**. Therefore, she will end up at a location **A** + (–**B**), or **A** – **B**. Note that –**B** has the same magnitude as **B** (30.0 m), but is in the opposite direction, $68^\circ\,(180^\circ - 112^\circ)$ south of east, as illustrated in Figure 5.14.

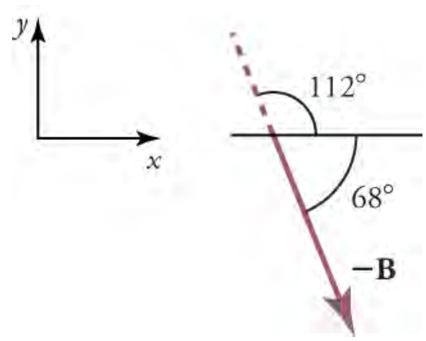

Figure 5.14 Vector –B represents traveling in the opposite direction of vector B.

We use graphical vector addition to find where the woman arrives **A** + (–**B**).

Solution

(1) To determine the location at which the woman arrives by accident, draw vectors **A** and –**B**.

(2) Place the vectors head to tail.

(3) Draw the resultant vector **R**.

(4) Use a ruler and protractor to measure the magnitude and direction of **R**.

These steps are demonstrated in Figure 5.15.

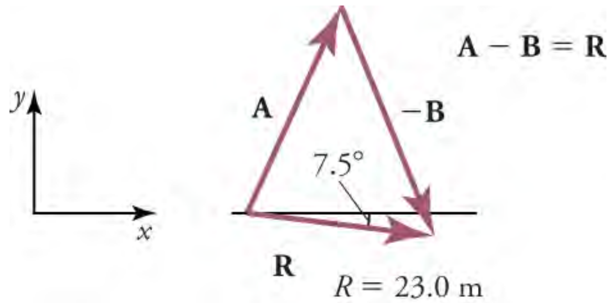

Figure 5.15 The vectors are placed head to tail.

In this case

$$R = 23.0 \text{ m} \tag{5.3}$$

and

$$\theta = 7.5°\text{south of east.} \tag{5.4}$$

Discussion

Because subtraction of a vector is the same as addition of the same vector with the opposite direction, the graphical method for subtracting vectors works the same as for adding vectors.

WORKED EXAMPLE

Adding Velocities: A Boat on a River

A boat attempts to travel straight across a river at a speed of 3.8 m/s. The river current flows at a speed v_{river} of 6.1 m/s to the right. What is the total velocity and direction of the boat? You can represent each meter per second of velocity as one centimeter of vector length in your drawing.

Strategy

We start by choosing a coordinate system with its x-axis parallel to the velocity of the river. Because the boat is directed straight toward the other shore, its velocity is perpendicular to the velocity of the river. We draw the two vectors, $\mathbf{v}_{\text{boat}}$ and $\mathbf{v}_{\text{river}}$, as shown in Figure 5.16.

Using the head-to-tail method, we draw the resulting total velocity vector from the tail of $\mathbf{v}_{\text{boat}}$ to the head of $\mathbf{v}_{\text{river}}$.

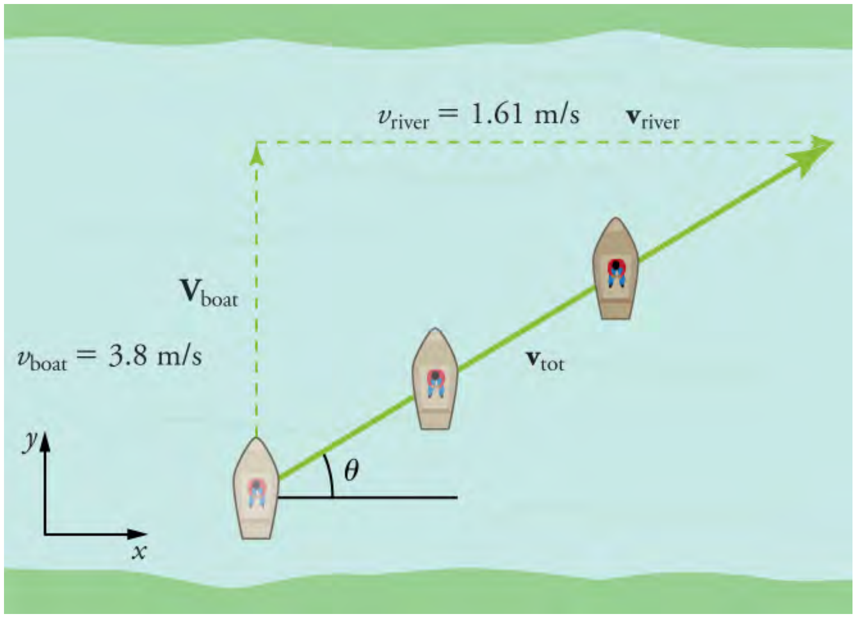

Figure 5.16 A boat attempts to travel across a river. What is the total velocity and direction of the boat?

Solution

By using a ruler, we find that the length of the resultant vector is 7.2 cm, which means that the magnitude of the total velocity is

$$v_{tot} = 7.2 \text{ m/s.} \tag{5.5}$$

By using a protractor to measure the angle, we find $\theta = 32.0^\circ$.

Discussion

If the velocity of the boat and river were equal, then the direction of the total velocity would have been 45°. However, since the velocity of the river is greater than that of the boat, the direction is less than 45° with respect to the shore, or *x* axis.

Practice Problems

1. Vector $\vec{A}$, having magnitude 2.5 m, pointing 37° south of east and vector $\vec{B}$ having magnitude 3.5 m, pointing 20° north of east are added. What is the magnitude of the resultant vector?
   a. 1.0 m
   b. 5.3 m
   c. 5.9 m
   d. 6.0 m

2. A person walks 32° north of west for 94 m and 35° east of south for 122 m. What is the magnitude of his displacement?
   a. 28 m
   b. 51 m
   c. 180 m
   d. 216 m

Virtual Physics

Vector Addition

In this simulation (https://archive.cnx.org/specials/d218bf9b-e50e-4d50-9a6c-b3db4dad0816/vector-addition/), you will experiment with adding vectors graphically. Click and drag the red vectors from the Grab One basket onto the graph in the middle of the screen. These red vectors can be rotated, stretched, or repositioned by clicking and dragging with your mouse. Check the Show Sum box to display the resultant vector (in green), which is the sum of all of the red vectors placed on the graph. To remove a red vector, drag it to the trash or click the Clear All button if you wish to start over. Notice that, if you click on any of the vectors, the $|\mathbf{R}|$ is its magnitude, θ is its direction with respect to the positive x-axis, $\mathbf{R_x}$ is its horizontal component, and R_y is its vertical component. You can check the resultant by lining up the vectors so that the head of the first vector touches the tail of the second. Continue until all of the vectors are aligned together head-to-tail. You will see that the resultant magnitude and angle is the same as the arrow drawn from the tail of the first vector to the head of the last vector. Rearrange the vectors in any order head-to-tail and compare. The resultant will always be the same.

Click to view content (https://archive.cnx.org/specials/d218bf9b-e50e-4d50-9a6c-b3db4dad0816/vector-addition/)

GRASP CHECK

True or False—The more long, red vectors you put on the graph, rotated in any direction, the greater the magnitude of the resultant green vector.

a. True
b. False

Check Your Understanding

3. While there is no single correct choice for the sign of axes, which of the following are conventionally considered positive?
   a. backward and to the left
   b. backward and to the right
   c. forward and to the right
   d. forward and to the left

4. True or False—A person walks 2 blocks east and 5 blocks north. Another person walks 5 blocks north and then two blocks east. The displacement of the first person will be more than the displacement of the second person.
   a. True
   b. False

5.2 Vector Addition and Subtraction: Analytical Methods

Section Learning Objectives

By the end of this section, you will be able to do the following:

- Define components of vectors
- Describe the analytical method of vector addition and subtraction
- Use the analytical method of vector addition and subtraction to solve problems

Section Key Terms

analytical method	component (of a two-dimensional vector)

Components of Vectors

For the **analytical method** of vector addition and subtraction, we use some simple geometry and trigonometry, instead of using a ruler and protractor as we did for graphical methods. However, the graphical method will still come in handy to visualize the problem by drawing vectors using the head-to-tail method. The analytical method is more accurate than the graphical method, which is limited by the precision of the drawing. For a refresher on the definitions of the sine, cosine, and tangent of an angle, see Figure 5.17.